# Grassmannians and their Chow rings

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## 1 Motivation: enumerative geometry

Our current goal is to use the machinery of intersection theory to solve problems in enumerative geometry. Some examples:

**Question 1.1.** *Given four general lines in $\mathbb{P}^3$, how many lines intersect all four?*

**Question 1.2.** *Given four general curves in $\mathbb{P}^3$ of degrees $d_1, \ldots, d_4$, how many lines intersect all four?*

**Question 1.3.** *Given two general twisted cubics in $\mathbb{P}^3$, how many chords do they have in common?*

**Question 1.4.** *Given four general quadric surfaces in $\mathbb{P}^3$, how many lines are tangent to all four?*

All of these questions concern the set of lines in $\mathbb{P}^3$ satisfying certain properties, so Grassmannians (specifically $\mathbb{G}(1,3)$) are a natural place to look for answers to these questions. Moreover, each of the four conditions appears to cut out a codimension-4 subvariety of the 4-dimensional variety $\mathbb{G}(1,3)$, so we can reasonably hope to obtain finite, nonzero answers. However, the word "general" is obviously needed in all four questions: for example, there are infinitely many lines meeting four given lines if all four happen to be equal, or even contained within a common plane.

Our general strategy will be as follows:

1. Find a moduli space $\mathcal{H}$ (e.g. a Grassmannian) for the objects we are interested in. This moduli space should ideally be smooth and projective.
2. Describe the Chow ring $A(\mathcal{H})$.
3. Find the classes $[Z_i] \in A(\mathcal{H})$, where $Z_i$ are the loci of objects with the given properties.
4. Calculate the intersection product.

*Notes for a talk given for David Eisenbud's reading course on intersection theory. Reference: Eisenbud and Harris, *3264 & All That: Intersection Theory in Algebraic Geometry*, first half of chapter 3, minus 3.2.(2-6), plus the first two sentences of 3.4.1.

5. Check that the $Z_i$ actually intersect generically transversely.

The last step is necessary because we care about the set-theoretic intersection, which may not agree with the intersection product. A priori, it could be that the dimension of the intersection is unexpectedly high, or that it contains points with multiplicity. But we would still know something. For example, when we solve Question 1.1, we will find that the intersection product is the class of two points. Here, even without worrying about technicalities (or the word "general"!), we will know that the locus of lines intersecting all four of our lines is either two distinct points, one point of multiplicity two, or a positive-dimensional variety. In particular, this implies that the answer is always nonzero.

On the other hand, if the intersection product turned out to be a negative number of points, then we would know that the intersection locus has greater-than-expected dimension, due to (e.g.) intersecting a $(-1)$-curve with itself. If it were zero, we would know that the intersection locus is either empty or infinite.

# 2 Introduction to Grassmannians

Definition: the Grassmannian is the moduli space of $k$-dimensional planes in $n$-dimensional space. This begs the question: affine or projective? We will mostly be working in projective space, but will use both. Let $G(k, n)$ or $G(k, V)$ denote the space of $k$-dimensional planes in the $n$-dimensional vector space $V$, and $\mathbb{G}(k, n)$ similarly in projective space. Of course, we have a natural isomorphism $G(k, n) \cong \mathbb{G}(k-1, n-1)$.

If $V$ is $n$-dimensional, there is also a natural duality isomorphism between $G(k, V)$ and $G(n-k, V^*)$. In particular, $G(1, n)$ and $G(n-1, n)$ are both isomorphic to $\mathbb{P}^n$. So the first "non-trivial" Grassmannian is $G(2, 4) = \mathbb{G}(1, 3)$. Today, we will briefly discuss Grassmannians in general, and then focus on $\mathbb{G}(1, 3)$.

## 2.1 Plücker embedding

We want Grassmannians to be smooth projective varieties, so we must find a way to identify $G(k, V)$ with a closed subvariety of some projective space. This is accomplished by the *Plücker embedding.* The Plücker embedding sends each $k$-plane $\Lambda \subset V$ to its $k$th exterior power $\bigwedge^k \Lambda \subset \bigwedge^k V$. Since $\bigwedge^k \Lambda$ is a one-dimensional space and $\bigwedge^k V$ is $\binom{n}{k}$-dimensional, this is a point in $\mathbb{P}^{\binom{n}{k}-1}$.

We claim that this map is actually injective. To prove this, let $v_1, \dots, v_k$ be a basis for $\Lambda$, so that $\bigwedge^k \Lambda$ is the span of $v_1 \wedge \cdots \wedge v_k$. Notice that this space is annihilated (as a subset of the exterior algebra $\bigwedge^* V$) by a vector $w \in V$ if and only if $w \in \Lambda$. So we can recover the data of $\Lambda$ from its top exterior power, as desired.

Fixing a basis $e_1, \dots, e_n$ for $V$, we can impose homogeneous coordinates on the target space $\mathbb{P}^{\binom{n}{k}-1}$ by choosing a basis of the form $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$. These pull back to a homogeneous coordinate system for $G(k,V)$. An easy calculation shows that the resulting *Plücker coordinates* $p_{i_1,\dots,i_k}$ are the $k \times k$ minors of (any choice of) a $k \times n$ matrix whose rows form a basis for $\Lambda$.

There are a few different ways to show that the image of the Plücker embedding is closed. (For example: it is the vanishing set of a certain family of $(n-k+1) \times (n-k+1)$ minors.) Since we will only care about $G(2,4) = \mathbb{G}(1,3)$ today, I'll state the result in this case: $\mathbb{G}(1,3)$ is a hypersurface of $\mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ cut out (in Plücker coordinates) by the equation $p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3} = 0$.

# 3 The Chow ring of $\mathbb{G}(1,3)$

We will now compute the Chow ring of $\mathbb{G}(1,3)$, using an affine stratification given by Schubert cells. Fix a complete flag $\mathcal{V} = (p \in L \subset H)$ in $\mathbb{P}^3$. Let $\Lambda$ denote a line in $\mathbb{P}^3$, so that $[\Lambda] \in \mathbb{G}(1,3)$. We will study the Grassmannian by classifying the different ways that $\Lambda$ can intersect $\mathcal{V}$.

(Picture goes here.) We define six *Schubert cycles*, which are closed subvarieties of $\mathbb{G}(1,3)$:

$$\Sigma_{0,0} = \text{the locus of all lines,} \tag{1}$$
$$\Sigma_{1,0} = \text{the locus of lines that intersect } L, \tag{2}$$
$$\Sigma_{2,0} = \text{the locus of lines that contain } p, \tag{3}$$
$$\Sigma_{1,1} = \text{the locus of lines contained in } H, \tag{4}$$
$$\Sigma_{2,1} = \text{the locus of lines contained in } H \text{ that contain } p, \text{ and} \tag{5}$$
$$\Sigma_{2,2} = \text{the locus consisting of } L. \tag{6}$$

To summarize, $\Sigma_{a,b}$ (for $0 \le b \le a \le 2$) denotes the locus of lines that intersect $\mathcal{V}$ at least $a$ dimensions before expected, and are contained in $V$ at least $b$ dimensions before expected. You can convince yourselves that $\Sigma_{a,b}$ is an irreducible subvariety of $\mathbb{G}(1,3)$ with codimension $a+b$.

Notation: for each $a,b$, we let $\Sigma^\circ_{a,b}$ denote $\Sigma_{a,b}$ minus all the $\Sigma_{a',b'}$ properly contained in it. (Each of these *Schubert cells* is open in its respective Schubert cycle.) Finally, the *Schubert class* $\sigma_{a,b}$ is the class of $\Sigma_{a,b}$ in $A^{a+b}(\mathbb{G}(1,3))$. (These do not depend on the choice of flag, since there is an automorphism of $\mathbb{P}^3$ taking any flag to any other.) When discussing all of these objects, we often drop the second subscript $b$ when it is 0.

We will now use these Schubert cells to study the Chow group of $\mathbb{G}(1,3)$.

**Lemma 3.1.** *The Schubert cells $\Sigma^\circ_{a,b}$ form an affine stratification of $\mathbb{G}(1,3)$: that is, they are all locally closed and irreducible, $\mathbb{G}(1,3)$ is their disjoint union, the closure $\overline{\Sigma^\circ_{a,b}} = \Sigma_{a,b}$ is a union of Schubert cells, and each $\Sigma^\circ_{a,b}$ is isomorphic to an affine space $\mathbb{A}^d$.*

*Proof.* Everything is clear but the last statement, which amounts to the uniqueness of reduced row echelon form in linear algebra. Recall that a line $\Lambda$ in $\mathbb{P}^3$ is equivalent to a 2-plane $\widetilde{\Lambda}$ in affine 4-space. Say $\widetilde{\Lambda}$ is the span of two vectors, which we write as row vectors of length 4. Then we can reduce the $2 \times 4$ matrix $\begin{bmatrix} - & v_1 & - \\ - & v_2 & - \end{bmatrix}$ to a unique reduced row echelon form, preserving the span of the rows. The RREF can take six possible forms, which correspond exactly to the six Schubert cells:

$$\Sigma_0^\circ \leftrightarrow \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \tag{7}$$

$$\Sigma_1^\circ \leftrightarrow \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \tag{8}$$

$$\Sigma_2^\circ \leftrightarrow \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{9}$$

$$\Sigma_{1,1}^\circ \leftrightarrow \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \tag{10}$$

$$\Sigma_{2,1}^\circ \leftrightarrow \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

$$\Sigma_{2,2}^\circ \leftrightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

The correspondence works by choosing the flag $\mathcal{V}$ as follows: the point $p$ is the 1-dimensional space spanned by the last coordinate; $L$ is the 2-dimensional space spanned by the last two; and $H$ is the 3-dimensional space spanned by the last 3. The six resulting varieties are visibly isomorphic to $\mathbb{A}^d$, where the dimension is respectively 4, 3, 2, 2, 1, and 0. $\square$

We are now ready to describe the ring structure of $A(\mathbb{G}(1,3))$.

**Theorem 3.2.** *The Chow ring of $\mathbb{G}(1,3)$ is generated by the six Schubert classes $\sigma_{a,b} \in A^{a+b}, 0 \leq b \leq a \leq 2$, as a free abelian group, and its multiplication is given by:*

$$\sigma_1^2 = \sigma_2 + \sigma_{1,1}; \tag{13}$$

$$\sigma_1\sigma_{1,1} = \sigma_1\sigma_2 = \sigma_{2,1}; \tag{14}$$

$$\sigma_1\sigma_{2,1} = \sigma_{2,2}; \tag{15}$$

$$\sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}; \tag{16}$$

$$\sigma_{1,1}\sigma_2 = 0. \tag{17}$$

*(Any two cycles whose codimensions add to more than 4 necessarily have trivial product.)*

*Proof.* The additive group structure follows from the fact that the $\Sigma_{a,b}^\circ$ form an affine stratification. Most of the products can be checked by very elementary means; that is, choose two different flags so that the relevant intersections are transverse, and compute the set of lines satisfying the given conditions. The identity $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$ is a bit more difficult. For this, we instead use the fact that $\sigma_1^2 = \alpha\sigma_2 + \beta\sigma_{1,1}$, then multiply both sides respectively by $\sigma_2$ and by $\sigma_{1,1}$ to conclude that $\alpha = \beta = 1$. $\square$

As a result of these computations, we can finally answer our first question, at least modulo the verification of generic transversality. The locus of lines meeting a given line is $\Sigma_1$, whose class is $\sigma_1 \in A^1(\mathbb{G}(1,3))$. The fourfold intersection product of this class is:

$$\begin{aligned} \sigma_1^4 &= (\sigma_{1,1} + \sigma_2)^2 && (18) \\ &= \sigma_{1,1}^2 + 2\sigma_{1,1}\sigma_2 + \sigma_2^2 && (19) \\ &= 2\sigma_{2,2}, && (20) \end{aligned}$$

which is the class of two points. So we expect (but do not yet truly know) that there should be exactly two lines intersecting each of four given general lines in $\mathbb{P}^3$.